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# The directed switching game on Lawrence oriented matroids

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## ABSTRACT

The main content of the note is a proof of the conjecture of Hamidoune and Las Vergnas on the directed switching game in the case of Lawrence oriented matroids.

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C.E. Shannon introduced the switching game for graphs circa 1960. The game has been generalized and solved for matroids by A. Lehman [1]. A directed switching game on graphs and oriented matroids was introduced by Y. O. Hamidoune and M. Las Vergnas in 1986 [2]. They have solved it for graphic and cographic oriented matroids, and stated as a conjecture that the classification of the oriented game is identical to the classification of its non-oriented version. This conjecture holds for the graphic and cographic cases, but remains open for more general classes of oriented matroids. In this note, we show that it holds for the class of Lawrence oriented matroids.

**Definition 1.** Let  $\mathcal{M}$  be an oriented matroid and  $e$  one of its elements. In the *directed switching game* on  $\mathcal{M}$ , Maker and Breaker alternately play by choosing an unplayed element of  $\mathcal{M}$  different from  $e$ , Maker signs it and Breaker deletes it. Maker wins the game if the final orientation of  $\mathcal{M}$  contains a positive circuit containing  $e$ .

In order to prove that the classification of the directed switching game is identical to the classification of the undirected game, it suffices to show

**Conjecture ([2]).** *If  $\mathcal{M}$  is the union of two disjoint bases, then the directed switching game on  $\mathcal{M}$  is winning for Maker playing first.*

**Definition 2.** The *Lawrence oriented matroid* defined by an  $n \times r$  matrix  $A = (a_{ij})$  with coefficients in  $\{-1, 1\}$  is the uniform oriented matroid of rank  $r$  on  $n$  elements such that the sign of an ordered basis  $(i_1, \dots, i_r)_<$  is given by

$$\chi(i_1, \dots, i_r) = \prod_{j=1}^r a_{j, i_j}.$$

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For more details, we refer the reader to Section 3.5 of [3] for chirotopes and Section 7.6 for Lawrence matroids (where they are called  $\Gamma$ ). Lawrence matroids are uniform and vectorial oriented matroids. The special case when all the coefficients of the matrix are 1 gives the alternating matroid (the oriented matroid of the cyclic polytope).

Let  $C = \{i_1, i_2, \dots, i_{r+1}\}_<$  be a circuit of a uniform matroid. For every element  $i$  of the circuit, the set  $C \setminus i$  is an ordered basis of the matroid. The relation between the signs  $C_{i_j}, C_{i_{j+1}}$  of two consecutive elements  $i_j, i_{j+1}$  of  $C$ , and the signs of the bases  $B_j = C \setminus i_j$  and  $B_{j+1} = C \setminus i_{j+1}$ , is given by

$$\chi(B_j) \cdot \chi(B_{j+1}) = -C(i_j) \cdot C(i_{j+1}).$$

It follows that in a Lawrence matroid the signature of  $C$  is given by  $C(i_1) = +$  and recursively by  $C(i_{j+1}) = -C(i_j) \cdot a_{j,i_j} \cdot a_{j,i_{j+1}}$  for  $1 \leq j \leq r$ .

**Theorem.** *The directed switching game on a Lawrence matroid of rank  $r$  and of order  $n$  is winning for Maker playing first if and only if  $n \geq 2r$ .*

**Proof.** If  $n < 2r$ , then Maker does not sign enough elements to create a circuit and then loses.

Suppose that  $n = 2r$  and let  $k$  be the initial element. A winning strategy for Maker will be to play  $a = \lceil \frac{k-1}{2} \rceil$  elements smaller than  $k$  and  $b = \lceil \frac{n-k}{2} \rceil$  elements bigger than  $k$ . Note that the relation  $a+b+1 = r+1$  is verified and that  $k$  in the end will correspond to the element  $c_{a+1}$  in the constructed circuit  $C = \{c_1, \dots, c_{r+1}\}$ .

Maker plays on the side (left or right) of  $k$  where an odd number, say  $2j+1$ , of elements is left. On this side, Maker chooses  $i$ , the closest element to  $k$ . In the case where the chosen element is smaller than  $k$ , this element will be  $c_{j+1}$ . In the other case, the element  $i$  will be  $c_{r+1-j}$ .

This permits Maker to sign  $i$  by  $-a_{j+1,i} \cdot a_{j+1,i'}$  in the first case and by  $-a_{r-j,i} \cdot a_{r-j,i'}$ , where  $i'$  is the previous element played on this side (possibly  $k$  if none). Of course, these rules are used to have at the end  $C(i) = C(i')$ , which implies that after  $r$  moves of Maker, the set of selected signed elements forms a positive circuit.

In the case  $n > 2r$ , Maker can use fictitious moves like in [2]. Maker will first select from the whole set a subset of  $2r$  elements containing  $k$ . Then Maker applies the previous strategy on this subset by choosing an element for Breaker in the case where Breaker plays outside the subset (these are the fictitious moves).  $\square$

## References

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